Exact Solution of Linear System of Fractional Differential Equations with Constant Coefficients

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative, the exact solution of linear system of fractional differential equations with constant coefficients is obtained. A new multiplication of fractional analytic functions plays an important role in this paper. In addition, we also provide some examples to illustrate the application of our results. In fact, our results are generalizations of these results in ordinary differential equations.**

Keywords: **Jumarie type of R-L fractional derivative, exact solution, linear system of fractional differential equations with constant coefficients, new multiplication, fractional analytic functions.**

I. INTRODUCTION

The history of fractional calculus is almost as long as the development of traditional calculus. In 1695, the concept of fractional derivative first appeared in a famous letter between L'Hospital and Leibniz. Many great mathematicians have further developed this field, such as Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, and Weyl. In the past few decades, fractional calculus has played a very important role in physics, electrical engineering, economics, biology, control theory, and other fields [1-7].

However, the definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie type of R-L fractional derivative [8-12]. Since the Jumarie type of R-L fractional derivative makes the derivative of constant function equal to zero, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie's modified R-L fractional derivative, we obtain the exact solution of linear system of fractional differential equations with constant coefficients. A new multiplication of fractional analytic functions plays an important role in this article. Moreover, two examples are provided to illustrate the application of our results. And our results are generalizations of these results in ordinary differential equations.

II. PRELIMINARIES

Firstly, the fractional calculus used in this paper and some important properties are introduced below.

Definition 2.1 ([13])**:** Assume that $0 < \alpha \le 1$, and t_0 is a real number. The Jumarie's modified R-L α-fractional derivative is defined by

$$\left({}_{t_0}D_t^{\alpha}\right)[f(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_0}^{t}\frac{f(x)-f(t_0)}{(t-x)^{\alpha}}dx, \tag{1}$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([14])**:** *If* α, β, t_0, c *are real numbers and* $\beta \ge \alpha > 0$, *then*

$$\left({}_{t_0}D_t^{\alpha}\right)\left[(t-t_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-t_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{t_0}D_t^{\alpha}\right)[c] = 0. \tag{3}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([15])**:** Assume that t, t_0, and a_k are real numbers for all k, $t_0 \in (a,b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}$ on some open interval containing t_0, then we say that $f_\alpha(t^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Definition 2.4 ([16])**:** If $0 < \alpha \le 1$, t_0 is a real number, and $f_\alpha(t^\alpha)$ and $g_\alpha(t^\alpha)$ are two α-fractional analytic functions defined on an interval containing t_0 ,

$$f_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}, \tag{4}$$

$$g_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}. \tag{5}$$

Then

$$\begin{aligned} & f_\alpha(t^\alpha) \otimes g_\alpha(t^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)(t-t_0)^{k\alpha}. \end{aligned} \tag{6}$$

In other words,

$$\begin{aligned} & f_\alpha(t^\alpha) \otimes g_\alpha(t^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{7}$$

Definition 2.5 ([17])**:** If $0 < \alpha \le 1$, and t is a real number. The α-fractional exponential function is defined by

$$E_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes k}. \tag{8}$$

In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes 2k}, \tag{9}$$

and

$$sin_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes(2k+1)}. \tag{10}$$

Theorem 2.6 ([17])**:** *If $0 < \alpha \le 1$ and p, q are two real numbers. Then*

$$E_\alpha((p+iq)t^\alpha) = E_\alpha(pt^\alpha) \otimes \left(cos_\alpha(qt^\alpha) + i sin_\alpha(qt^\alpha)\right). \tag{11}$$

III. RESULTS AND EXAMPLES

In this section, the main results including the exact solution of linear system of fractional differential equations with constant coefficients are obtained. On the other hand, two examples are provided to illustrate the application of our results.

Definition 3.1: Let $0 < \alpha \le 1$, n be a positive integer. The matrix form of linear system of fractional differential equations with constant coefficients is

$$\left({}_{0}D_t^{\alpha}\right)[x_\alpha(t^\alpha)] = A\, x_\alpha(t^\alpha), \tag{12}$$

where $x_\alpha(t^\alpha) = \begin{bmatrix} x_\alpha^1(t^\alpha) \\ x_\alpha^2(t^\alpha) \\ \vdots \\ x_\alpha^n(t^\alpha) \end{bmatrix}$ and A is an $n \times n$ constant matrix.

Definition 3.2: If $0 < \alpha \le 1$, and A is an $n \times n$ constant matrix. Then we define

$$E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} A^k t^{k\alpha} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \left(\frac{1}{\Gamma(\alpha+1)} t^\alpha\right)^{\otimes k}. \tag{13}$$

Theorem 3.3: *If $0 < \alpha \le 1$, and A is a $n \times n$ matrix. Then the linear system of α-fractional differential equations with constant coefficients*

$$\left({}_0D_t^\alpha\right)[x_\alpha(t^\alpha)] = A\, x_\alpha(t^\alpha) \tag{14}$$

has the solution

$$x_\alpha(t^\alpha) = E_\alpha(At^\alpha)\, x_\alpha(0). \tag{15}$$

Proof If $x_\alpha(t^\alpha) = E_\alpha(At^\alpha)\, x_\alpha(0)$, then

$$\begin{aligned} &\left({}_0D_t^\alpha\right)[x_\alpha(t^\alpha)] \\ &= \left({}_0D_t^\alpha\right)[E_\alpha(At^\alpha)\, x_\alpha(0)] \\ &= \left({}_0D_t^\alpha\right)\left[\sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} A^k t^{k\alpha}\; x_\alpha(0)\right] \\ &= \sum_{k=0}^{\infty} \left({}_0D_t^\alpha\right)\left[\frac{1}{\Gamma(k\alpha+1)} A^k t^{k\alpha}\right] x_\alpha(0) \\ &= A\left[\sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} A^k t^{k\alpha}\right] x_\alpha(0) \\ &= AE_\alpha(At^\alpha)\, x_\alpha(0) \\ &= A\, x_\alpha(t^\alpha). \end{aligned}$$

Therefore, the desired result holds. Q.e.d.

Definition 3.4: Suppose that $0 < \alpha \le 1$, and n is a positive integer. If $\varphi_\alpha^1(t^\alpha)$, $\varphi_\alpha^2(t^\alpha)$,⋯, $\varphi_\alpha^n(t^\alpha)$ are linearly independent solutions of $\left({}_0D_t^\alpha\right)[x_\alpha(t^\alpha)] = A\, x_\alpha(t^\alpha)$, then the matrix

$$\Phi_\alpha(t^\alpha) = [\varphi_\alpha^1(t^\alpha), \varphi_\alpha^2(t^\alpha), \cdots, \varphi_\alpha^n(t^\alpha)] \tag{16}$$

is called a fundamental matrix solution of $\left({}_0D_t^\alpha\right)[x_\alpha(t^\alpha)] = A\, x_\alpha(t^\alpha)$.

Theorem 3.5: *Let $0 < \alpha \le 1$. Then $\Phi_\alpha(t^\alpha)$ is a fundamental matrix solution of $\left({}_0D_t^\alpha\right)[x_\alpha(t^\alpha)] = A\, x_\alpha(t^\alpha)$ if and only if $\det \Phi_\alpha(t^\alpha) \ne 0$. Moreover, if $\det \Phi_\alpha(t_0^\alpha) \ne 0$ for some real number t_0, then $\det \Phi_\alpha(t^\alpha) \ne 0$ for all t.*

Theorem 3.6: *Let $0 < \alpha \le 1$. If the matrix A has eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ with linearly independent eigenvectors $v_1, v_2, \cdots, v_n$ respectively. Then*

$$\Phi_\alpha(t^\alpha) = [E_\alpha(\lambda_1 t^\alpha)v_1, E_\alpha(\lambda_2 t^\alpha)v_2, \cdots, E_\alpha(\lambda_n t^\alpha)v_n] \tag{17}$$

is a fundamental matrix solution of $\left({}_0D_t^\alpha\right)[x_\alpha(t^\alpha)] = A\, x_\alpha(t^\alpha)$.

Proof Since for all $i = 1, 2, \cdots, n$,

$$\begin{aligned} &\left({}_0D_t^\alpha\right)[E_\alpha(\lambda_i t^\alpha)v_i] \\ &= \lambda_i E_\alpha(\lambda_i t^\alpha)v_i \\ &= A\, E_\alpha(\lambda_i t^\alpha)v_i\,, \end{aligned} \tag{18}$$

and $\det \Phi_\alpha(0) = [v_1, v_2, \cdots, v_n] \ne 0$, it follows from Theorem 3.5 that the desired result holds. Q.e.d.

Theorem 3.7: *If* $0<\alpha\leq 1$. *Then* $\Phi_\alpha(t^\alpha)=E_\alpha(At^\alpha)$ *is a fundamental matrix solution of* $\left({}_0D_t^\alpha\right)[x_\alpha(t^\alpha)]=A\,x_\alpha(t^\alpha)$.

Proof Since

$$\begin{aligned}&\left({}_0D_t^\alpha\right)[\,\Phi_\alpha(t^\alpha)]\\&=\left({}_0D_t^\alpha\right)[E_\alpha(At^\alpha)]\\&=AE_\alpha(At^\alpha)\\&=A\,\Phi_\alpha(t^\alpha).\end{aligned}\tag{19}$$

It follows that the desired result holds. Q.e.d.

Theorem 3.8: *Let* $0<\alpha\leq 1$. *If* $\Phi_\alpha(t^\alpha)$ *is a fundamental matrix solution of* $\left({}_0D_t^\alpha\right)[x_\alpha(t^\alpha)]=A\,x_\alpha(t^\alpha)$, *then*

$$E_\alpha(At^\alpha)=\Phi_\alpha(t^\alpha)\Phi_\alpha^{-1}(0).\tag{20}$$

Proof Since $E_\alpha(At^\alpha)$ and $\Phi_\alpha(t^\alpha)$ are fundamental matrix solutions of $\left({}_0D_t^\alpha\right)[x_\alpha(t^\alpha)]=A\,x_\alpha(t^\alpha)$, it follows that

$$E_\alpha(At^\alpha)=\Phi_\alpha(t^\alpha)C,\tag{21}$$

where C is a nonsingular constant matrix. If $t=0$, then $I=\Phi_\alpha(0)C$. Thus, $C=\Phi_\alpha^{-1}(0)$. Hence, the desired result holds.

Q.e.d.

Theorem 3.9: *If* $0<\alpha\leq 1$, A,B,P *are* $n\times n$ *matrices, and* P *is a nonsingular matrix. Then*

$$E_\alpha((A+B)t^\alpha)=E_\alpha(At^\alpha)\otimes E_\alpha(Bt^\alpha),\text{ if } AB=BA.\tag{22}$$

$$[E_\alpha(At^\alpha)]^{\otimes-1}=E_\alpha(-At^\alpha),\tag{23}$$

$$E_\alpha(P^{-1}APt^\alpha)=P^{-1}E_\alpha(At^\alpha)P.\tag{24}$$

Proof Since $AB=BA$, it follows that

$$\begin{aligned}&E_\alpha(At^\alpha)\otimes E_\alpha(Bt^\alpha)\\&=\sum_{k=0}^{\infty}\frac{1}{k!}A^k\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{1}{k!}B^k\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes k}\\&=\left(I+A\frac{1}{\Gamma(\alpha+1)}t^\alpha+\frac{1}{2!}A^2\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes 2}+\cdots\right)\otimes\left(I+B\frac{1}{\Gamma(\alpha+1)}t^\alpha+\frac{1}{2!}B^2\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes 2}+\cdots\right)\\&=I+(A+B)\frac{1}{\Gamma(\alpha+1)}t^\alpha+\frac{1}{2!}(A^2+2AB+B^2)\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes 2}+\cdots\\&=I+(A+B)\frac{1}{\Gamma(\alpha+1)}t^\alpha+\frac{1}{2!}(A+B)^2\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes 2}+\cdots\\&=\sum_{k=0}^{\infty}\frac{1}{k!}(A+B)^k\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes k}\\&=E_\alpha((A+B)t^\alpha).\end{aligned}\tag{25}$$

On the other hand, since A and $-A$ are commutative,

$$E_\alpha(At^\alpha)\otimes E_\alpha(-At^\alpha)=E_\alpha\big((A-A)t^\alpha\big)=I.\tag{26}$$

Thus,

$$E_\alpha(-At^\alpha)=[E_\alpha(At^\alpha)]^{\otimes-1}.$$

Finally,

$$\begin{aligned}&E_\alpha(P^{-1}APt^\alpha)\\&=\sum_{k=0}^{\infty}\frac{1}{k!}(P^{-1}AP)^k\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes k}\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} P^{-1} A^k P \left(\frac{1}{\Gamma(\alpha+1)} t^{\alpha}\right)^{\otimes k}$$

$$= P^{-1} \left(\sum_{k=0}^{\infty} \frac{1}{k!} A^k \left(\frac{1}{\Gamma(\alpha+1)} t^{\alpha}\right)^{\otimes k}\right) P$$

$$= P^{-1} E_{\alpha}(A t^{\alpha}) P \ . \tag{27}$$

Example 3.10: Let $0 < \alpha \leq 1$.Find the solution of the initial-value problem of linear system of α-fractional differential equations with constant coefficients

$$\left({}_0D_t^{\alpha}\right) \begin{bmatrix} x_{\alpha}^1(t^{\alpha}) \\ x_{\alpha}^2(t^{\alpha}) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_{\alpha}^1(t^{\alpha}) \\ x_{\alpha}^2(t^{\alpha}) \end{bmatrix}, \quad \begin{bmatrix} x_{\alpha}^1(0) \\ x_{\alpha}^2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{28}$$

Solution Since $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ are commutative, it follows from Theorem 3.9 that

$$E_{\alpha}(A t^{\alpha})$$

$$= E_{\alpha}\left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} t^{\alpha}\right)$$

$$= E_{\alpha}\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} t^{\alpha}\right) \otimes E_{\alpha}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t^{\alpha}\right)$$

$$= \begin{bmatrix} E_{\alpha}(2t^{\alpha}) & 0 \\ 0 & E_{\alpha}(2t^{\alpha}) \end{bmatrix} \otimes \left(I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{\Gamma(\alpha+1)} t^{\alpha} + \frac{1}{2!} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 \left(\frac{1}{\Gamma(\alpha+1)} t^{\alpha}\right)^{\otimes 2} + \cdots\right)$$

$$= \begin{bmatrix} E_{\alpha}(2t^{\alpha}) & 0 \\ 0 & E_{\alpha}(2t^{\alpha}) \end{bmatrix} \otimes \begin{bmatrix} 1 & \frac{1}{\Gamma(\alpha+1)} t^{\alpha} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} E_{\alpha}(2t^{\alpha}) & \frac{1}{\Gamma(\alpha+1)} t^{\alpha} \otimes E_{\alpha}(2t^{\alpha}) \\ 0 & E_{\alpha}(2t^{\alpha}) \end{bmatrix}. \tag{29}$$

And hence, by Theorem 3.3, the solution is

$$\begin{bmatrix} x_{\alpha}^1(t^{\alpha}) \\ x_{\alpha}^2(t^{\alpha}) \end{bmatrix}$$

$$= \begin{bmatrix} E_{\alpha}(2t^{\alpha}) & \frac{1}{\Gamma(\alpha+1)} t^{\alpha} \otimes E_{\alpha}(2t^{\alpha}) \\ 0 & E_{\alpha}(2t^{\alpha}) \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} E_{\alpha}(2t^{\alpha}) \otimes \left(2 - \frac{1}{\Gamma(\alpha+1)} t^{\alpha}\right) \\ -E_{\alpha}(2t^{\alpha}) \end{bmatrix}. \tag{30}$$

That is,

$$x_{\alpha}^1(t^{\alpha}) = E_{\alpha}(2t^{\alpha}) \otimes \left(2 - \frac{1}{\Gamma(\alpha+1)} t^{\alpha}\right), \tag{31}$$

$$x_{\alpha}^2(t^{\alpha}) = -E_{\alpha}(2t^{\alpha}). \tag{32}$$

Example 3.11: If $0 < \alpha \leq 1$.Find the solution of the initial-value problem of linear system of α-fractional differential equations with constant coefficients

$$\left({}_0D_t^{\alpha}\right) \begin{bmatrix} x_{\alpha}^1(t^{\alpha}) \\ x_{\alpha}^2(t^{\alpha}) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} x_{\alpha}^1(t^{\alpha}) \\ x_{\alpha}^2(t^{\alpha}) \end{bmatrix}, \quad \begin{bmatrix} x_{\alpha}^1(0) \\ x_{\alpha}^2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}. \tag{33}$$

Solution Let $A = \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix}$, then $\det(\lambda I - A) = 0$ implies that the eigenvalues of A are $\lambda_1 = 3 + 5i$, $\lambda_2 = 3 - 5i$. We can easily obtain the eigenvector of λ_1 is $v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$, and the eigenvector of λ_2 is $v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$. Therefore, by Theorem 3.6 we obtain a fundamental matrix solution of this linear system of α-fractional differential equations,

$$\Phi_\alpha(t^\alpha)=\begin{bmatrix} E_\alpha((3+5i)t^\alpha) & iE_\alpha((3-5i)t^\alpha) \\ iE_\alpha((3+5i)t^\alpha) & E_\alpha((3-5i)t^\alpha) \end{bmatrix}. \tag{34}$$

Using Theorem 2.6 and Theorem 3.8 yields

$$\begin{aligned}
&E_\alpha(At^\alpha)\\
&=\Phi_\alpha(t^\alpha)\Phi_\alpha^{-1}(0)\\
&=\begin{bmatrix} E_\alpha((3+5i)t^\alpha) & iE_\alpha((3-5i)t^\alpha) \\ iE_\alpha((3+5i)t^\alpha) & E_\alpha((3-5i)t^\alpha) \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^{-1}\\
&=\begin{bmatrix} E_\alpha((3+5i)t^\alpha) & iE_\alpha((3-5i)t^\alpha) \\ iE_\alpha((3+5i)t^\alpha) & E_\alpha((3-5i)t^\alpha) \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\\
&=\frac{1}{2}\begin{bmatrix} E_\alpha((3+5i)t^\alpha)+E_\alpha((3-5i)t^\alpha) & -iE_\alpha\left((3+5i)t^\alpha\right)+iE_\alpha((3-5i)t^\alpha) \\ iE_\alpha((3+5i)t^\alpha)-iE_\alpha((3-5i)t^\alpha) & E_\alpha((3+5i)t^\alpha)+E_\alpha((3-5i)t^\alpha) \end{bmatrix}\\
&=\begin{bmatrix} E_\alpha(3t^\alpha)\otimes cos_\alpha(5t^\alpha) & E_\alpha(3t^\alpha)\otimes sin_\alpha(5t^\alpha) \\ -E_\alpha(3t^\alpha)\otimes sin_\alpha(5t^\alpha) & E_\alpha(3t^\alpha)\otimes cos_\alpha(5t^\alpha) \end{bmatrix}.
\end{aligned} \tag{35}$$

Thus, by Theorem 3.3, the solution is

$$\begin{aligned}
&\begin{bmatrix} x_\alpha^1(t^\alpha) \\ x_\alpha^2(t^\alpha) \end{bmatrix}\\
&=E_\alpha(At^\alpha)\,x_\alpha(0)\\
&=\begin{bmatrix} E_\alpha(3t^\alpha)\otimes cos_\alpha(5t^\alpha) & E_\alpha(3t^\alpha)\otimes sin_\alpha(5t^\alpha) \\ -E_\alpha(3t^\alpha)\otimes sin_\alpha(5t^\alpha) & E_\alpha(3t^\alpha)\otimes cos_\alpha(5t^\alpha) \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}\\
&=\begin{bmatrix} E_\alpha(3t^\alpha)\otimes\left(-cos_\alpha(5t^\alpha)+3sin_\alpha(5t^\alpha)\right) \\ E_\alpha(3t^\alpha)\otimes\left(3cos_\alpha(5t^\alpha)+sin_\alpha(5t^\alpha)\right) \end{bmatrix}.
\end{aligned} \tag{36}$$

That is,

$$x_\alpha^1(t^\alpha)=E_\alpha(3t^\alpha)\otimes\left(-cos_\alpha(5t^\alpha)+3sin_\alpha(5t^\alpha)\right), \tag{37}$$

$$x_\alpha^2(t^\alpha)=E_\alpha(3t^\alpha)\otimes\left(3cos_\alpha(5t^\alpha)+sin_\alpha(5t^\alpha)\right). \tag{38}$$

IV. CONCLUSION

In this paper, based on Jumarie’s modified R-L fractional derivative, we obtain the exact solution of linear system of fractional differential equations with constant coefficients. A new multiplication of fractional analytic functions plays an important role in this article. On the other hand, we give some examples to illustrate the application of our results. In fact, the results we obtained are generalizations of these results in ordinary differential equations. In the future, we will continue to study the problems in fractional differential equations and applied mathematics.

REFERENCES

[1] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[2] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[3] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp. 41-45, 2016.

[4] R. L. Magin, Fractional calculus in bioengineering, 13th International Carpathian Control Conference, 2012.

[5] R. Hilfer (ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[6] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[7] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[8] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, Inc., 1974.

[9] S. Das, Functional Fractional Calculus, 2nd ed. Springer-Verlag, 2011.

[10] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, USA, 1993.

[11] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[12] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[13] C. -H. Yu, Fractional derivative of arbitrary real power of fractional analytic function, International Journal of Novel Research in Engineering and Science, vol. 9, no. 1, pp. 9-13, 2022.

[14] U. Ghosh, S. Sengupta, S. Sarkar and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, vol. 3, no. 2, pp. 32-38, 2015.

[15] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.

[16] C. -H. Yu, A study on arc length of nondifferentiable curves, Research Inventy: International Journal of Engineering and Science, vol. 12, no. 4, pp. 18-23, 2022.

[17] C. -H. Yu, Linear systems of fractional differential equations, International Journal of Novel Research in Interdisciplinary Studies, vol. 9, no. 5, pp. 1-6, 2022.